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Herausgeber: Der Rektor

Exponential stability for second order evolutionary problems.

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MATH-AN-03-2014

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April 14, 2015

Abstract. We study the exponential stability of evolutionary equations. The focus is laid on second order problems and we provide a way to rewrite them as a suitable first order evolutionary equation, for which the stability can be proved by using frequency domain methods. The problem class under consideration is broad enough to cover integro-differential equations, delay-equations and classical evolution equations within a unified framework.

Keywords and phrases: Exponential stability, hyperbolic problems, evolutionary equations, integro-differential equations, frequency domain methods, delay.

Mathematics subject classification 2010: 35B30; 35B40; 35G15; 47N20

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1 Introduction

Evolutionary equations, as they were first introduced by Picard ([12, 13, 15]), consist of a first order differential equation on \mathbb{R} as the time-line

$$\partial_0 v + Au = f,$$

where ∂_0 denotes the derivative with respect to time and A is a suitable closed linear operator on a Hilbert space (frequently a block-operator-matrix with spatial differential operators as entries). The function u and v are the unknowns, while f is a given source term. This simple equation is completed by a so-called linear material law linking u and v :

$$v = \mathcal{M}u,$$

where \mathcal{M} is an operator acting in time and space. Thus, the differential equation becomes

$$(\partial_0 \mathcal{M} + A)u = f \tag{1}$$

a so-called evolutionary problem. The operator sum on the left-hand side will be established in a suitable Hilbert space and thus, the well-posedness of (1) relies on the bounded invertibility of that operator sum. For doing so, one establishes the time-derivative ∂_0 as a normal boundedly invertible operator in a suitable exponentially weighted L_2 -space. With the spectral representation of this normal operator at hand, one specifies the operator \mathcal{M} as to be an analytic operator-valued function of ∂_0^{-1} . Then \mathcal{M} enjoys the property that it is causal due to the Theorem of Paley and Wiener (see e.g. [19, Theorem 19.2]). Although the requirement of analyticity seems to be very strong, these operators cover a broad class of possible space-time operators like convolutions with suitable kernels naturally arising in the study of integro-differential equations (see e.g. [22]), translations with respect to time as they occur in delay equations (see [10]) as well as fractional derivatives (see [14]). Thus, the setting of evolutionary equations provides a unified framework for a broad class of partial differential equations. We note that the causality of \mathcal{M} also yields the causality of the solution operator $(\partial_0 \mathcal{M} + A)^{-1}$ of our evolutionary problem (1), which can be seen as a characterizing property of evolutionary processes.

After establishing the well-posedness of (1), one is interested in qualitative properties of the solution u . A first property, which can be discussed, is the asymptotic behaviour of u , especially the question of exponential stability. The study of stability for differential equations goes back to Lyapunov and a lot of approaches has been developed to tackle this question over the last decades. We just like to mention some classical results for evolution equations, using the framework of strongly continuous semigroups, like Datko's Lemma [6] or the Theorem of Gearhart-Prüss [9, 17] (see also [7, Chapter V] for the asymptotics of semigroups). Unfortunately, these results are not applicable to evolutionary equations. The main reason for that is that the solution u of (1) is not continuous, unless the right-hand side f is regular, so that point-wise estimates for the solution u (and this is how exponential stability is usually defined) do not make any sense. Hence, we need to introduce a more general notion of exponential stability for that class of problems. This was done by the author in [20, 21] (see also Subsection 2.2 in this article), where also sufficient conditions on the material law \mathcal{M} to obtain exponential stability were derived.

The main purpose of this article is to study the exponential stability of evolutionary problems of second order in time and space, i.e. to equations of the form

$$(\partial_0^2 \mathcal{M} + C^* C) u = f, \quad (2)$$

where C is a densely defined closed linear operator, which is assumed to be boundedly invertible. For doing so, we need to rewrite the above problem as an evolutionary equation of first order in time. As it turns out there are several ways to do this yielding a family of new material law operators $(\mathcal{M}_d)_{d>0}$, such that (2) can be written as

$$\left(\partial_0 \mathcal{M}_d + \begin{pmatrix} 0 & C^* \\ -C & 0 \end{pmatrix} \right) \begin{pmatrix} \partial_0 u + du \\ Cu \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}, \quad (3)$$

for each $d > 0$. The goal is now to state sufficient conditions for the original operator \mathcal{M} , such that there is $d > 0$ for which the exponential stability of the equivalent problem (3) can be derived. This is the main objective of Subsection 3.1.

The article is structured as follows. Section 2 is devoted to the framework of evolutionary equations. In this section we recall the definition of the time-derivative ∂_0 and of linear material laws. Moreover, we introduce the notion of exponential stability for evolutionary equations and provide a characterization result using Frequency Domain Methods (Theorem 2.7). As already indicated above, Subsection 3.1 deals with the exponential stability of (2) or equivalently of (3) and provides sufficient constraints on the material law \mathcal{M} , which yield the exponential stability. In Subsection 3.2 we focus on special material law operators \mathcal{M} , namely convolutions with a kernel k . We derive sufficient conditions on the kernel k , such that the corresponding integro-differential equations becomes exponentially stable. Exponential stability of hyperbolic integro-differential equations was studied for instance in [2] and [18] (where in [18] also polynomial stability was addressed). We show that our conditions on the kernel k , which – in contrast to the aforementioned references – is operator-valued, are weaker than the ones imposed in [2, 18]. Finally, in Section 4, we treat a concrete example of a wave equation with convolution integral and time delay. This problem was recently studied in [1] by using semigroup techniques and constructing a suitable energy for which the exponential decay was shown. We will see that this problem is covered by our abstract considerations and hence, the exponential stability follows. Moreover, our approach allows to relax the assumptions on the kernel of the convolution integral.

Throughout, all Hilbert spaces are assumed to be complex. Their inner products are denoted by $\langle \cdot | \cdot \rangle$, which are linear in the second and conjugate linear in the first argument and the induced norms are denoted by $|\cdot|$.

2 Evolutionary problems

In this section we recall the notion of evolutionary problems, as they were first introduced in [12], [13, Chapter 6] (see also [15] for a survey). We begin by introducing the time-derivative in an exponentially weighted L_2 -space in the first subsection. The second subsection is devoted to the well-posedness and exponential stability of evolutionary problems. The main theorem in this subsection (Theorem 2.7) characterizes the exponential stability of an evolutionary problem by point-wise properties of the unitarily equivalent multiplication operators (such

methods are frequently referred to as Frequency Domain Methods). We remark that we generalize the notion of evolutionary problems in the sense that we do not impose monotonicity constraints on the operators involved. Throughout, let H be a Hilbert space.

2.1 The time-derivative

We introduce the time-derivative as a boundedly invertible operator on an exponentially weighted L_2 -space. This idea first appears in [16]. For the proofs of the forthcoming results, we refer to [12, 10, 13]. For $\varrho \in \mathbb{R}$ we consider the following Hilbert space

$$H_\varrho(\mathbb{R}; H) := \left\{ f : \mathbb{R} \rightarrow H \mid f \text{ measurable, } \int_{\mathbb{R}} |f(t)|^2 e^{-2\varrho t} dt < \infty \right\}$$

equipped with the inner product

$$\langle f | g \rangle_{H_\varrho(\mathbb{R}; H)} := \int_{\mathbb{R}} \langle f(t) | g(t) \rangle_H e^{-2\varrho t} dt \quad (f, g \in H_\varrho(\mathbb{R}; H)).$$

We note that for $\varrho = 0$ this is nothing but the usual $L_2(\mathbb{R}; H)$. Moreover we define the operators

$$\begin{aligned} e^{-\varrho m} : H_\varrho(\mathbb{R}; H) &\rightarrow L_2(\mathbb{R}; H) \\ f &\mapsto (t \mapsto e^{-\varrho t} f(t)) \end{aligned}$$

which are obviously unitary. We denote the derivative on $L_2(\mathbb{R}; H)$ with maximal domain by ∂ , i.e.

$$\begin{aligned} \partial : H^1(\mathbb{R}; H) &\subseteq L_2(\mathbb{R}; H) \rightarrow L_2(\mathbb{R}; H) \\ f &\mapsto f', \end{aligned}$$

where $H^1(\mathbb{R}; H)$ is the classical Sobolev-space of L_2 -functions whose distributional derivative also belongs to L_2 . It is well-known that this operator is skew-selfadjoint with $\sigma(\partial) = i\mathbb{R}$ (see e.g. [11, Example 3.14]). Moreover, it is well-known that ∂ is unitarily equivalent to the multiplication operator im on $L_2(\mathbb{R}; H)$ with maximal domain, i.e.

$$D(\text{im}) := \{f \in L_2(\mathbb{R}; H) \mid (\text{im})f = (t \mapsto itf(t)) \in L_2(\mathbb{R}; H)\},$$

where the unitary transformation is given by the Fourier-transform, defined as the unitary extension of

$$\mathcal{F}|_{L_1(\mathbb{R}; H) \cap L_2(\mathbb{R}; H)} : L_1(\mathbb{R}; H) \cap L_2(\mathbb{R}; H) \subseteq L_2(\mathbb{R}; H) \rightarrow L_2(\mathbb{R}; H)$$

with

$$(\mathcal{F}f)(t) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ist} f(s) ds \quad (f \in L_1(\mathbb{R}; H) \cap L_2(\mathbb{R}; H), t \in \mathbb{R}).$$

In other words we have

$$\partial = \mathcal{F}^* (\text{im}) \mathcal{F}.$$

Using now the unitary operators $e^{-\varrho m}$ we define the derivative $\partial_{0,\varrho}$ on $H_\varrho(\mathbb{R}; H)$ by¹

$$\partial_{0,\varrho} := (e^{-\varrho m})^{-1} \partial e^{-\varrho m} + \varrho.$$

Indeed, this definition yields

$$(\partial_{0,\varrho}\phi)(t) = e^{\varrho t} (\phi'(t)e^{-\varrho t} - \varrho\phi(t)e^{-\varrho t}) + \varrho\phi(t) = \phi'(t) \quad (t \in \mathbb{R})$$

for each $\phi \in C_c^\infty(\mathbb{R}; H)$ – the space of arbitrarily differentiable functions on \mathbb{R} with values in H and compact support. As an immediate consequence of this definition we obtain that

$$\sigma(\partial_{0,\varrho}) = \varrho + i\mathbb{R},$$

which in particular yields that $\partial_{0,\varrho}$ is boundedly invertible if and only if $\varrho \neq 0$. Moreover, the spectrum is continuous spectrum. The inverse $\partial_{0,\varrho}^{-1}$ is given by

$$(\partial_{0,\varrho}^{-1}f)(t) = \begin{cases} \int_{-\infty}^t f(s) \, ds & \text{if } \varrho > 0, \\ -\int_t^\infty f(s) \, ds & \text{if } \varrho < 0. \end{cases}$$

Thus, $\varrho > 0$ corresponds to the so-called causal² case, while $\varrho < 0$ gives the anticausality of $\partial_{0,\varrho}^{-1}$. We further note that also $\partial_{0,\varrho}$ is unitarily equivalent to a multiplication operator on $L_2(\mathbb{R}; H)$. The unitary mapping is the so-called Fourier-Laplace transform given as

$$\mathcal{L}_\varrho := \mathcal{F}e^{-\varrho m} : H_\varrho(\mathbb{R}; H) \rightarrow L_2(\mathbb{R}; H).$$

Indeed, we obtain that

$$\begin{aligned} \mathcal{L}_\varrho \partial_{0,\varrho} \mathcal{L}_\varrho^* &= \mathcal{F}e^{-\varrho m} \left((e^{-\varrho m})^{-1} \partial e^{-\varrho m} + \varrho \right) (e^{-\varrho m})^{-1} \mathcal{F}^* \\ &= \mathcal{F} \partial \mathcal{F}^* + \varrho \\ &= \text{im} + \varrho. \end{aligned}$$

2.2 Well-posedness and exponential stability

Throughout, let $A : D(A) \subseteq H \rightarrow H$ be a densely defined closed linear operator. We define now, what we mean by a linear material law.

Definition. Let $\Omega \subseteq \mathbb{C}$ open such that³ $\mathbb{C}_{\Re z > \mu} \setminus \{0\} \subseteq [\Omega]^{-1} = \{z^{-1} \mid z \in \Omega\}$ for some $\mu \in \mathbb{R}$. A *linear material law* is an analytic mapping $M : \Omega \rightarrow L(H)$.

¹The equality of operators especially implies the equality of their domains. Thus, the domain of $\partial_{0,\varrho}$ is given by the natural domain of the operator $(e^{-\varrho m})^{-1} \partial e^{-\varrho m} + \varrho$, which is

$$\{u \in H_\varrho(\mathbb{R}; H) \mid e^{-\varrho m}u \in H^1(\mathbb{R}; H)\}.$$

²Roughly speaking, causality means that the image at time t just depends on the values of the pre-image up to the same time t , while anticausality means that it just depends on the values of the pre-image beginning at time t . For a precise definition of causality, we refer to Remark 2.3 (a).

³For $\mu \in \mathbb{R}$ we set

$$\begin{aligned} \mathbb{C}_{\Re z \geq \mu} &:= \left\{ z \in \mathbb{C} \mid \Re z \geq \mu \right\}, \\ \mathbb{C}_{\Im z \geq \mu} &:= \left\{ z \in \mathbb{C} \mid \Im z \geq \mu \right\}. \end{aligned}$$

As multiplication operators will play an important role in the framework of evolutionary problems, we introduce them in the following definition.

Definition. For $t \in \mathbb{R}$ let $T(t)$ be a linear operator on H . Then we set

$$\begin{aligned} T(\mathbf{m}) : D(T(\mathbf{m})) &\subseteq L_2(\mathbb{R}; H) \rightarrow L_2(\mathbb{R}; H) \\ f &\mapsto (t \mapsto T(t)f(t)), \end{aligned}$$

where

$$D(T(\mathbf{m})) := \{f \in L_2(\mathbb{R}; H) \mid f(t) \in D(T(t)) \text{ for a.e. } t \in \mathbb{R}, (t \mapsto T(t)f(t)) \in L_2(\mathbb{R}; H)\}.$$

With the help of this notion we are able to define so-called evolutionary problems.

Definition. Let M be a linear material law. We associate the multiplication operator $M\left(\frac{1}{\mathbf{i}m + \varrho}\right)$ on $L_2(\mathbb{R}; H)$ for ϱ large enough, i.e.

$$\left(M\left(\frac{1}{\mathbf{i}m + \varrho}\right)f\right)(t) := M\left(\frac{1}{\mathbf{i}t + \varrho}\right)f(t),$$

where $f \in L_2(\mathbb{R}; H)$ such that $(t \mapsto M\left(\frac{1}{\mathbf{i}t + \varrho}\right)f(t)) \in L_2(\mathbb{R}; H)$. Moreover, we consider its unitarily equivalent operator

$$M(\partial_{0,\varrho}^{-1}) := \mathcal{L}_\varrho^* M\left(\frac{1}{\mathbf{i}m + \varrho}\right) \mathcal{L}_\varrho$$

with its natural domain (cp. Footnote 1). An *evolutionary problem* is an equation in $H_\varrho(\mathbb{R}; H)$ of the form

$$\left(\partial_{0,\varrho} M(\partial_{0,\varrho}^{-1}) + A\right)u = f, \quad (4)$$

where we identify A with its canonical extension to $H_\varrho(\mathbb{R}; H)$ given by $(Au)(t) := A(u(t))$ for almost every $t \in \mathbb{R}$ and $u \in H_\varrho(\mathbb{R}; H)$ such that $u(t) \in D(A)$ for almost every $t \in \mathbb{R}$ and $(t \mapsto A(u(t))) \in H_\varrho(\mathbb{R}; H)$.

Let us illustrate the class of evolutionary equations by some examples.

Example 2.1.

- (a) Let $A = \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix} : D(C) \times D(C^*) \subseteq H_0 \oplus H_1 \rightarrow H_0 \oplus H_1$, where $C : D(C) \subseteq H_0 \rightarrow H_1$ is a densely defined closed linear operator⁴ between two Hilbert spaces H_0, H_1 . By setting $M(z) := M_0 + zM_1$ for operators $M_0, M_1 \in L(H)$ with $H := H_0 \oplus H_1$ we cover a class of parabolic, hyperbolic and elliptic problems. Indeed, if $M_0 = \begin{pmatrix} M_{00} & 0 \\ 0 & 0 \end{pmatrix}, M_1 = \begin{pmatrix} 0 & 0 \\ 0 & M_{11} \end{pmatrix}$ the problem reads as

$$\left(\partial_{0,\varrho} \left(M_0 + \partial_{0,\varrho}^{-1} M_1\right) + A\right)u = f.$$

⁴A typical example could be $C = \text{grad}$, where $D(C) = H_0^1(\Omega), H_0 = L_2(\Omega)$ and $H_1 = L_2(\Omega)^n$. Then $C^* = -\text{div}$, the divergence on L_2 . But also $C = \text{curl}$ is possible, which allows the treatment of Maxwell's equations.

In particular, letting $f = \begin{pmatrix} h \\ 0 \end{pmatrix} \in H_{\varrho,0}(\mathbb{R}; H)$ and setting $u = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$ we read off that

$$\begin{aligned}\partial_{0,\varrho} M_{00} u_0 - C^* u_1 &= f, \\ M_{11} u_1 + C u_0 &= 0.\end{aligned}$$

Thus, assuming that M_{11} is boundedly invertible, the second equation reads as $u_1 = -M_{11}^{-1} C u_0$ and thus, we indeed end up with an equation of parabolic type for u_0

$$\partial_{0,\varrho} M_{00} u_0 + C^* M_{11}^{-1} C u_0 = f.$$

If $M_0 = \begin{pmatrix} M_{00} & 0 \\ 0 & M_{01} \end{pmatrix}$ and $M_1 = 0$ we get

$$\begin{aligned}\partial_{0,\varrho} M_{00} u_0 - C^* u_1 &= f, \\ \partial_{0,\varrho} M_{01} u_1 + C u_0 &= 0.\end{aligned}$$

Again, assuming that M_{01} is boundedly invertible, we obtain $u_1 = -\partial_{0,\varrho}^{-1} M_{01}^{-1} C u_0$ and thus, the first equation reads as

$$\partial_{0,\varrho} M_{00} u_0 + C^* \partial_{0,\varrho}^{-1} M_{01}^{-1} C u_0 = f.$$

Differentiation yields

$$\partial_{0,\varrho}^2 M_{00} u_0 + C^* M_{01}^{-1} C u_0 = \partial_{0,\varrho} f,$$

which gives an equation of hyperbolic type. Finally, choosing $M_0 = 0$ and $M_1 = \begin{pmatrix} M_{10} & 0 \\ 0 & M_{11} \end{pmatrix}$ we end up with an elliptic type problem of the form

$$\begin{aligned}M_{10} u_0 - C^* u_1 &= f, \\ M_{11} u_1 + C u_0 &= 0,\end{aligned}$$

which may be rewritten as

$$M_{10} u_0 + C^* M_{11}^{-1} C u_0 = f.$$

Also, problems of mixed type are treatable, i.e. equations which are hyperbolic on one part of the domain, parabolic on another one and elliptic on a third part (see e.g. [15, Example 2.43], [24, p.8]). It should be noted that in all previous examples, M_0 and M_1 are also allowed to have non-vanishing off-diagonal entries. Moreover, in the abstract setting of evolutionary equations, there is no need to assume that the block structures of A and M_0 and M_1 are comparable. This allows the treatment of even more general differential equations.

- (b) Let $H = L_2(\Omega)$ for some $\Omega \subseteq \mathbb{R}^n$ and $k : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ be a measurable integrable function. Setting $M(z) = \sqrt{2\pi} \widehat{k}(-iz^{-1})$, where \widehat{k} denotes the Fourier transform of k , we end up with an evolutionary problem of the form

$$\partial_{0,\varrho} k * u + Au = f,$$

which is an integro-differential equation. For a detailed study of integro-differential equations within the framework of evolutionary problems we refer to [22]. A concrete example is also treated in Section 4.

(c) Setting $M(z) = z^{-\alpha}$ for some $\alpha \in]0, 1[$ we get

$$\partial_{0,\varrho}\partial_{0,\varrho}^{-\alpha}u + Au = \partial_{0,\varrho}^{1-\alpha}u + Au = f,$$

which covers a class of fractional differential equations. For more details and a more complicated examples we refer to [14].

Lemma 2.2. *Let $\varrho \in \mathbb{R}$. We consider a continuous mapping $L : \{it + \varrho \mid t \in \mathbb{R}\} \rightarrow L(H)$. For $t \in \mathbb{R}$ we define*

$$\begin{aligned} T(it + \varrho) : D(A) \subseteq H &\rightarrow H \\ x &\mapsto (L(it + \varrho) + A)x. \end{aligned}$$

We consider the corresponding multiplication operators $T(\text{im} + \varrho)$ and $L(\text{im} + \varrho)$ on $L_2(\mathbb{R}; H)$. Then⁵

$$T(\text{im} + \varrho) = \overline{L(\text{im} + \varrho) + A},$$

where we identify A with its canonical extension to $L_2(\mathbb{R}; H)$.

Proof. We first prove that $T(\text{im} + \varrho)$ is closed. For doing so, let $(f_n)_{n \in \mathbb{N}}$ in $D(T(\text{im} + \varrho))$ with $f_n \rightarrow f$ and $T(\text{im} + \varrho)f_n \rightarrow g$ in $L_2(\mathbb{R}; H)$. By passing to a suitable subsequence, we may assume without loss of generality, that $f_n(t) \rightarrow f(t)$ and $(T(\text{im} + \varrho)f_n)(t) \rightarrow g(t)$ for almost every $t \in \mathbb{R}$. As $L(it + \varrho) \in L(H)$ for each $t \in \mathbb{R}$, we derive that

$$L(it + \varrho)f_n(t) \rightarrow L(it + \varrho)f(t)$$

for almost every $t \in \mathbb{R}$. Consequently

$$Af_n(t) = T(it + \varrho)f_n(t) - L(it + \varrho)f_n(t) \rightarrow g(t) - L(it + \varrho)f(t)$$

for almost every $t \in \mathbb{R}$ and hence, by the closedness of A we obtain $f(t) \in D(A)$ and $L(it + \varrho)f(t) + Af(t) = g(t)$ almost everywhere. This shows $f \in D(T(\text{im} + \varrho))$ and $T(\text{im} + \varrho)f = g$ and thus, $T(\text{im} + \varrho)$ is closed. Since trivially

$$L(\text{im} + \varrho) + A \subseteq T(\text{im} + \varrho)$$

we derive

$$\overline{L(\text{im} + \varrho) + A} \subseteq T(\text{im} + \varrho).$$

For showing the converse inclusion, let $f \in D(T(\text{im} + \varrho))$. For $n \in \mathbb{N}$ we define $f_n := \chi_{[-n, n]}(\text{im})f$ by $f_n(x) := \chi_{[-n, n]}(x)f(x)$ for $x \in \mathbb{R}$. We estimate

$$\int_{\mathbb{R}} |L(it + \varrho)f_n(t)|^2 dt = \int_{-n}^n |L(it + \varrho)f(t)|^2 dt \leq \sup_{t \in [-n, n]} \|L(it + \varrho)\|^2 |f|_{L_2(\mathbb{R}; H)}^2,$$

⁵Note that the natural domain of the operator $L(\text{im} + \varrho) + A$ is given by

$$\begin{aligned} &D(L(\text{im} + \varrho)) \cap D(A) \\ &= \{f \in L_2(\mathbb{R}; H) \mid f(t) \in D(A) \text{ for a.e. } t \in \mathbb{R}, (t \mapsto L(it + \varrho)f(t)) \in L_2(\mathbb{R}; H), (t \mapsto Af(t)) \in L_2(\mathbb{R}; H)\}, \end{aligned}$$

which is in general a proper subset of

$$D(T(\text{im} + \varrho)) = \{f \in L_2(\mathbb{R}; H) \mid f(t) \in D(A) \text{ for a.e. } t \in \mathbb{R}, (t \mapsto L(it + \varrho)f(t) + Af(t)) \in L_2(\mathbb{R}; H)\}.$$

showing that $f_n \in D(L(\text{im} + \varrho))$. Since clearly $f_n \in D(T(\text{im} + \varrho))$ we obtain that $f_n \in D(L(\text{im} + \varrho) + A)$. Now, since $f_n \rightarrow f$ as well as $(L(\text{im} + \varrho) + A)f_n = T(\text{im} + \varrho)f_n \rightarrow T(\text{im} + \varrho)f$ in $L_2(\mathbb{R}; H)$ by dominated convergence, we conclude $f \in D(\overline{L(\text{im} + \varrho) + A})$. \square

The latter lemma implies that the operator $\overline{\partial_{0,\varrho}M(\partial_{0,\varrho}^{-1}) + A}$ is unitarily equivalent (via the Fourier-Laplace transformation \mathcal{L}_ϱ) to the multiplication operator $T(\text{im} + \varrho)$, where $T(z) = zM(z^{-1}) + A$. Indeed, for $L(z) := zM(z^{-1})$ we get that

$$\mathcal{L}_\varrho \left(\overline{\partial_{0,\varrho}M(\partial_{0,\varrho}^{-1}) + A} \right) \mathcal{L}_\varrho^* = \overline{\mathcal{L}_\varrho \partial_{0,\varrho}M(\partial_{0,\varrho}^{-1}) \mathcal{L}_\varrho^* + A} = \overline{L(\text{im} + \varrho) + A} = T(\text{im} + \varrho). \quad (5)$$

Hence, the study of evolutionary equations is equivalent to the study of multiplication operators. This observation allows us to define well-posedness of an evolutionary problem in terms of the function T .

Definition. Let M be a linear material law. We call the associated evolutionary problem

$$\left(\partial_{0,\varrho}M(\partial_{0,\varrho}^{-1}) + A \right) u = f$$

well-posed, if there exists $\varrho_0 \in \mathbb{R}$ such that for each $z \in \mathbb{C}_{\Re > \varrho_0} \setminus \{0\}$ the operator $zM(z^{-1}) + A$ is boundedly invertible with $\sup_{z \in \mathbb{C}_{\Re > \varrho_0} \setminus \{0\}} \left\| (zM(z^{-1}) + A)^{-1} \right\| < \infty$. The infimum over all such numbers ϱ_0 is called the growth bound of the evolutionary problem and we denote it by $\omega_0(M, A)$.

Remark 2.3.

(a) We note that for a well-posed evolutionary problem we have that

$$\overline{\partial_{0,\varrho}M(\partial_{0,\varrho}^{-1}) + A}$$

is boundedly invertible for each $\varrho > \varrho_0$ due to (5). Moreover, by a Paley-Wiener-type argument (see e.g. [19, Theorem 19.2]), we obtain that the solution operator $S_\varrho := \left(\overline{\partial_{0,\varrho}M(\partial_{0,\varrho}^{-1}) + A} \right)^{-1}$ is forward causal, in the sense that

$$\chi_{]-\infty, a]}(\mathbf{m}) S_\varrho = \chi_{]-\infty, a]}(\mathbf{m}) S_\varrho \chi_{]-\infty, a]}(\mathbf{m}).$$

For more details we refer to [12, 13].

(b) For the evolutionary problems originally treated in [12], the operator A was assumed to be skew-selfadjoint, while the material law⁶ $M : B(r, r) \rightarrow L(H)$ was assumed to satisfy a positive definiteness constraint of the form⁷

$$\forall z \in B(r, r) : \Re z^{-1}M(z) \geq c > 0.$$

Clearly, these assumptions yield the well-posedness of the evolutionary problem in the above sense.

⁶For $z_0 \in \mathbb{C}$ and $r > 0$ we define $B(z_0, r) := \{z \in \mathbb{C} \mid |z - z_0| < r\}$. Likewise $B[z_0, r] := \{z \in \mathbb{C} \mid |z - z_0| \leq r\} = \overline{B(z_0, r)}$.

⁷For an operator $T \in L(H)$ we denote by $\Re T$ the selfadjoint operator $\frac{1}{2}(T + T^*)$.

- (c) Evolutionary problems may be written as an abstract operator equation of the form $(B + A)u = f$, where $B := \partial_{0,\varrho}M(\partial_{0,\varrho}^{-1})$. Such abstract problems were studied in a Banach space setting by da Prato and Grisvard [5] and the results were applied to differential equations of hyperbolic and parabolic type. However, these results are not applicable here, since the desired spectral properties of the operators involved are not met. Indeed, in [5] the operators have to verify, besides other spectral properties, a condition of Hille-Yosida type, which needs not to be satisfied for our choice of A and B .

In [8, Chapter 5] Favini and Yagi studied so-called degenerated differential equations on Banach spaces, which are of the form $(TM - L)u = f$, where T, M and L are operators on some Banach space. Moreover, the resolvent sets of T and M should contain certain logarithmic regions, while L is boundedly invertible and certain compatibility conditions for T, M and L are required. In our case this would correspond to $T = \partial_{0,\varrho}$, $M = M(\partial_{0,\varrho}^{-1})$ and $L = -A$, which in general do not satisfy these assumptions.

Moreover, we obtain that the solution operator S_ϱ does not depend on the particular choice of the parameter ϱ as the following proposition shows. Thus, we usually will omit the index ϱ in $\partial_{0,\varrho}$ and just write ∂_0 instead.

Proposition 2.4 ([20, Lemma 3.6]). *Let $\varrho, \varrho' \in \mathbb{R}$ with $\varrho' > \varrho$ and set $U := \{z \in \mathbb{C} \mid \Re z \in [\varrho, \varrho']\}$. Moreover, let $S : U \rightarrow L(H)$ be continuous, bounded and analytic on \mathring{U} and $f \in H_\varrho(\mathbb{R}; H) \cap H_{\varrho'}(\mathbb{R}; H)$. Then*

$$(\mathcal{L}_\varrho^* S(\text{im} + \varrho) \mathcal{L}_\varrho f)(t) = (\mathcal{L}_{\varrho'}^* S(\text{im} + \varrho') \mathcal{L}_{\varrho'} f)(t)$$

for almost every $t \in \mathbb{R}$.

We are now ready to introduce the notion of an exponentially stable evolutionary problem as it was defined in [20, 21].

Definition. A well-posed evolutionary problem is called *exponentially stable with stability rate $\varrho_1 > 0$* , if for each $0 \leq \nu < \varrho_1$ and $\varrho > \omega_0(M, A)$ and $f \in H_{-\nu}(\mathbb{R}; H) \cap H_\varrho(\mathbb{R}; H)$ we have that

$$\left(\overline{\partial_0 M(\partial_0^{-1}) + A} \right)^{-1} f \in H_{-\nu}(\mathbb{R}; H).$$

Remark 2.5.

- (a) We note that we cannot use the standard notion of exponential stability as it is used for instance in semigroup theory. There, one usually requires point-wise estimates for the solution u of the form $|u(t)| \leq M e^{-\omega t}$ for some $M, \omega > 0$ and each $t \in \mathbb{R}_{\geq 0}$. The main problem is that we do not have any regularizing property of our solution operator $\left(\overline{\partial_0 M(\partial_0^{-1}) + A} \right)^{-1}$ allowing to get continuous solutions. Thus, pointwise estimates cannot be used in our framework. Indeed, choosing for instance $M(\partial_0^{-1}) := \partial_0^{-1}$ we end up with a purely algebraic equation given by

$$(1 + A)u = f,$$

where we cannot expect continuity of the solution u , unless our right-hand side f is more regular than just square integrable. However, the notion of exponential stability as introduced above yields a classical point-wise estimate of the solution, if the right-hand side f is regular enough, for example an element in the domain of ∂_0 (see [20, Remark 3.2 (a)]).

- (b) We further remark that exponential stability is not just the requirement that we can solve the evolutionary problem for negative ϱ . The main problem is, that we need to preserve the causality of the solution operator, which is a typical property for positive but not for negative ϱ (compare $\partial_{0,\varrho}^{-1}$ in dependence of ϱ). That is why we need to define the exponential stability in terms of the causal solution operator, which is guaranteed by the choice $\varrho > \omega_0(M, A)$ in the latter definition. Indeed, consider the simple case $A = 0$ and $M(\partial_0^{-1}) = 1$. Then the corresponding evolutionary problem reads as

$$\partial_{0,\varrho} u = f$$

and we have $\omega(M, A) = \omega(1, 0) = 0$. This problem is solvable for positive and negative ϱ , yielding however, different solutions. Choosing for instance $f = \chi_{[0,1]}$ we get

$$\begin{aligned} u(t) &= \begin{cases} \int_{-\infty}^t f(s) \, ds & \text{if } \varrho > 0, \\ -\int_t^{\infty} f(s) \, ds & \text{if } \varrho < 0 \end{cases} \\ &= \begin{cases} t\chi_{[0,1]}(t) + \chi_{]1,\infty[}(t) & \text{if } \varrho > 0, \\ -(\chi_{]-\infty,0[}(t) + (1-t)\chi_{[0,1]}(t)) & \text{if } \varrho < 0 \end{cases} \end{aligned}$$

for $t \in \mathbb{R}$, which shows that even if f decays exponentially (it is even compactly supported) the causal solution (corresponding to positive ϱ) is not exponentially decaying.

In the subsequent theorem we will give a characterization of exponential stability in terms of the resolvents of $zM(z^{-1}) + A$ for $z \in \mathbb{C}_{\Re > -\varrho_1} \setminus \{0\}$. For doing so, we need the following auxiliary result.

Theorem 2.6 ([25]). *Let $S : L_2(\mathbb{R}_{\geq 0}; H) \rightarrow L_2(\mathbb{R}_{\geq 0}; H)$ be a bounded, shift-invariant linear operator. Then there exists a uniquely determined bounded and analytic function $N : \mathbb{C}_{\Re > 0} \rightarrow L(H)$ such that*

$$(\mathcal{L}_{\Re z} S f)(\Im z) = N(z) (\mathcal{L}_{\Re z} f)(\Im z)$$

for each $f \in L_2(\mathbb{R}_{\geq 0}; H)$ and every $z \in \mathbb{C}_{\Re > 0}$.

Now we are ready to state our characterization result.

Theorem 2.7. *Let $M : \mathbb{C} \setminus B[-r, r] \rightarrow L(H)$ be analytic and $0 < \varrho_1 < \frac{1}{2r}$. We assume that the evolutionary problem*

$$(\partial_0 M(\partial_0^{-1}) + A) u = f$$

is well-posed. Then the following statements are equivalent:

- (i) *The evolutionary problem $(\partial_0 M(\partial_0^{-1}) + A) u = f$ is exponentially stable with stability rate ϱ_1 ,*
- (ii) *For each $z \in \mathbb{C}_{\Re > -\varrho_1} \setminus \{0\}$ we have $0 \in \varrho(zM(z^{-1}) + A)$ and the function*

$$\begin{aligned} \mathbb{C}_{\Re > -\varrho_1} \setminus \{0\} &\rightarrow L(H) \\ z &\mapsto (zM(z^{-1}) + A)^{-1} \end{aligned}$$

is bounded.

Proof. Since our evolutionary problem is assumed to be well-posed, there is $\varrho_0 \in \mathbb{R}$ such that

$$\mathbb{C}_{\Re z > \varrho_0} \setminus \{0\} \ni z \mapsto (zM(z^{-1}) + A)^{-1} \in L(H)$$

is bounded and analytic.

(i) \Rightarrow (ii): The proof is done in 3 steps.

Step 1: We show that the operator

$$S := e^{\varrho_1 m} \left(\overline{\partial_0 M(\partial_0^{-1}) + A} \right)^{-1} e^{-\varrho_1 m} : L_2(\mathbb{R}_{\geq 0}; H) \rightarrow L_2(\mathbb{R}_{\geq 0}; H)$$

satisfies the assumptions of Theorem 2.6 and thus, there is $N : \mathbb{C}_{\Re z > 0} \rightarrow L(H)$ analytic and bounded such that

$$(\mathcal{L}_{\Re z} S f)(\Im z) = N(z) (\mathcal{L}_{\Re z} f)(\Im z)$$

for each $f \in L_2(\mathbb{R}_{\geq 0}; H)$ and every $z \in \mathbb{C}_{\Re z > 0}$.

Indeed, S is well-defined since for $f \in L_2(\mathbb{R}_{\geq 0}; H)$ it follows that $e^{-\varrho_1 m} f \in \bigcap_{\varrho \geq -\varrho_1} H_\varrho(\mathbb{R}_{\geq 0}; H)$, and thus, by assumption and the causality of $\left(\overline{\partial_0 M(\partial_0^{-1}) + A} \right)^{-1}$, we obtain that $S f \in L_2(\mathbb{R}_{\geq 0}; H)$. Moreover S is closed. Indeed, let $(f_n)_{n \in \mathbb{N}}$ be a sequence in $L_2(\mathbb{R}_{\geq 0}; H)$ such that $f_n \rightarrow f$ and $S f_n \rightarrow g$ in $L_2(\mathbb{R}_{\geq 0}; H)$ for some $f, g \in L_2(\mathbb{R}_{\geq 0}; H)$. Consequently $e^{-\varrho_1 m} f_n \rightarrow e^{-\varrho_1 m} f$ in $H_\varrho(\mathbb{R}_{\geq 0}; H)$ for each $\varrho \geq -\varrho_1$. By the boundedness of $\left(\overline{\partial_0 M(\partial_0^{-1}) + A} \right)^{-1}$ on $H_\varrho(\mathbb{R}_{\geq 0}; H)$ for $\varrho > \varrho_0$ we derive that

$$\left(\overline{\partial_0 M(\partial_0^{-1}) + A} \right)^{-1} e^{-\varrho_1 m} f_n \rightarrow \left(\overline{\partial_0 M(\partial_0^{-1}) + A} \right)^{-1} e^{-\varrho_1 m} f$$

in $H_\varrho(\mathbb{R}_{\geq 0}; H)$ and hence, $S f_n \rightarrow e^{\varrho_1 m} \left(\overline{\partial_0 M(\partial_0^{-1}) + A} \right)^{-1} e^{-\varrho_1 m} f = S f$ in $H_{\varrho+\varrho_1}(\mathbb{R}_{\geq 0}; H)$. As $L_2(\mathbb{R}_{\geq 0}; H) \hookrightarrow H_\varrho(\mathbb{R}_{\geq 0}; H)$ for each $\varrho \geq 0$, we derive that $g = S f$ and thus, S is closed. Hence, according to the closed graph theorem, we get that S is bounded. Since S is obviously translation invariant, by Theorem 2.6 there exists a unique analytic and bounded function $N : \mathbb{C}_{\Re z > 0} \rightarrow L(H)$ such that

$$(\mathcal{L}_{\Re z} S f)(\Im z) = N(z) (\mathcal{L}_{\Re z} f)(\Im z)$$

for each $f \in L_2(\mathbb{R}_{\geq 0}; H)$ and every $z \in \mathbb{C}_{\Re z > 0}$.

Step 2: We show that $N(z + \varrho_1) = (zM(z^{-1}) + A)^{-1}$ for each $z \in \mathbb{C}_{\Re z > \varrho_0}$.

Since for $\varrho > \varrho_0 + \varrho_1$ we have that (cp. (5))

$$\begin{aligned} (\mathcal{L}_\varrho S f)(t) &= \left(\mathcal{L}_\varrho e^{\varrho_1 m} \left(\overline{\partial_0 M(\partial_0^{-1}) + A} \right)^{-1} e^{-\varrho_1 m} f \right)(t) \\ &= \left(\mathcal{L}_{\varrho-\varrho_1} \left(\overline{\partial_0 M(\partial_0^{-1}) + A} \right)^{-1} e^{-\varrho_1 m} f \right)(t) \\ &= \left((it + \varrho - \varrho_1) M \left(\frac{1}{it + \varrho - \varrho_1} \right) + A \right)^{-1} (\mathcal{L}_{\varrho-\varrho_1} e^{-\varrho_1 m} f)(t), \\ &= \left((it + \varrho - \varrho_1) M \left(\frac{1}{it + \varrho - \varrho_1} \right) + A \right)^{-1} (\mathcal{L}_\varrho f)(t) \end{aligned}$$

for every $f \in L_2(\mathbb{R}_{\geq 0}; H)$, we derive that

$$N(z + \varrho_1) = (zM(z^{-1}) + A)^{-1}$$

for each $z \in \mathbb{C}_{\Re > \varrho_0}$.

Step 3: We show (ii), i.e. $0 \in \varrho(zM(z^{-1}) + A)$ for each $z \in \mathbb{C}_{\Re > -\varrho_1} \setminus \{0\}$.

We consider

$$\Omega := \{z \in \mathbb{C}_{\Re > -\varrho_1} \setminus \{0\} \mid 0 \in \varrho(zM(z^{-1}) + A)\} \subseteq \mathbb{C}_{\Re > -\varrho_1} \setminus \{0\}.$$

We first prove that Ω is open. For doing so let $z' \in \Omega$. Then for $z \in \mathbb{C}_{\Re > -\varrho_1} \setminus \{0\}$ we compute

$$\begin{aligned} zM(z^{-1}) + A &= zM(z^{-1}) - z'M(z'^{-1}) + z'M(z'^{-1}) + A \\ &= \left((zM(z^{-1}) - z'M(z'^{-1})) (z'M(z'^{-1}) + A)^{-1} + 1 \right) (z'M(z'^{-1}) + A). \end{aligned} \quad (6)$$

Due to the continuity of M there exists $\delta > 0$ such that for $|z - z'| < \delta$ we have that

$$\left\| (zM(z^{-1}) - z'M(z'^{-1})) (z'M(z'^{-1}) + A)^{-1} \right\| < 1.$$

Hence, by the Neumann series, $0 \in \varrho(zM(z^{-1}) + A)$ for each $|z - z'| < \delta$, showing that Ω is open.

We consider now the component C of $\varrho_0 + 1 \in \Omega$ in Ω . This component is open, since Ω is open. Moreover, due to the identity theorem, we obtain that $N(z + \varrho_1) = (zM(z^{-1}) + A)^{-1}$ for each $z \in C$. We will show that C is also closed in $\mathbb{C}_{\Re > -\varrho_1} \setminus \{0\}$. For doing so, let $(z_n)_{n \in \mathbb{N}}$ be a sequence in C converging to some $z \in \mathbb{C}_{\Re > -\varrho_1} \setminus \{0\}$. Since

$$\sup_{n \in \mathbb{N}} \left\| (z_n M(z_n^{-1}) + A)^{-1} \right\| = \sup_{n \in \mathbb{N}} \|N(z_n + \varrho_1)\| < \infty,$$

we obtain by (6) (replace z' by z_n) that $z \in \Omega$. Since Ω is open we find $\varepsilon > 0$ with $B(z, \varepsilon) \subseteq \Omega$. Moreover, there is $n \in \mathbb{N}$ with $z_n \in B(z, \varepsilon)$ showing that z and z_n belong to the same component C . Hence, C is clopen in $\mathbb{C}_{\Re > -\varrho_1} \setminus \{0\}$ and thus, $C = \mathbb{C}_{\Re > -\varrho_1} \setminus \{0\}$. The latter gives $\Omega = \mathbb{C}_{\Re > -\varrho_1} \setminus \{0\}$ and by the identity theorem we conclude

$$(zM(z^{-1}) + A)^{-1} = N(z + \varrho_1) \in L(H)$$

for each $z \in \mathbb{C}_{\Re > -\varrho_1} \setminus \{0\}$.

(ii) \Rightarrow (i): By assumption, the function

$$\begin{aligned} S : \mathbb{C}_{\Re > -\varrho_1} \setminus \{0\} &\rightarrow L(H) \\ z &\mapsto (zM(z^{-1}) + A)^{-1} \end{aligned}$$

is analytic and bounded and hence, its singularity in 0 is removable. We denote its analytic extension to $\mathbb{C}_{\Re > -\varrho_1}$ again by S . Consequently, for each $\varrho > -\varrho_0$ the multiplication operator $S(\text{im} + \varrho)$ is bounded. Its inverse is given by the multiplication $T(\text{im} + \varrho)$, where $T(z) := (zM(z^{-1}) + A)$. Hence, using Lemma 2.2, we obtain for each $\varrho > -\varrho_1$

$$S(\text{im} + \varrho) = \left(\overline{(\text{im} + \varrho) M \left(\frac{1}{\text{im} + \varrho} \right) + A} \right)^{-1}.$$

Let now $f \in H_{-\nu}(\mathbb{R}; H) \cap H_{\varrho}(\mathbb{R}; H)$ for some $0 \leq \nu < -\varrho_1$ and $\varrho > \omega_0(M, A)$. Then, by Proposition 2.4

$$\left(\overline{\partial_{0,\varrho} M(\partial_{0,\varrho}^{-1}) + A} \right)^{-1} f = \mathcal{L}_{\varrho}^* S(\text{im} + \varrho) \mathcal{L}_{\varrho} f = \mathcal{L}_{-\nu}^* S(\text{im} - \nu) \mathcal{L}_{-\nu} f \in H_{-\nu}(\mathbb{R}; H). \quad \square$$

As immediate consequences of the latter theorem we obtain the following two corollaries.

Corollary 2.8. *Let $M : \mathbb{C} \setminus B[-r, r] \rightarrow L(H)$ be analytic and assume that the corresponding evolutionary problem (4) is well-posed. Then it is exponentially stable with stability rate $0 < \varrho_1 < \frac{1}{2r}$ if and only if $\omega_0(M, A) \leq -\varrho_1$.*

Proof. According to Theorem 2.7, the evolutionary problem is exponentially stable with stability rate ϱ_1 if and only if the function

$$\mathbb{C}_{\Re > -\varrho_1} \setminus \{0\} \ni z \mapsto (zM(z^{-1}) + A)^{-1} \in L(H)$$

is bounded, which is nothing as to say that $\omega_0(M, A) \leq -\varrho_1$. \square

Corollary 2.9. *Let $M : \mathbb{C} \setminus B[-r, r] \rightarrow L(H)$ be analytic and assume that the evolutionary problem*

$$(\partial_0 M(\partial_0^{-1}) + A) u = f$$

is well-posed and exponentially stable with stability rate $0 < \varrho_1 < \frac{1}{2r}$. Let

$$C := \sup_{z \in \mathbb{C}_{\Re > -\varrho_1} \setminus \{0\}} \left\| (zM(z^{-1}) + A)^{-1} \right\|,$$

which is finite according to Theorem 2.7. Let $N : \mathbb{C} \setminus B[-r, r] \rightarrow L(H)$ be analytic and bounded such that

$$\|N\|_{\infty} := \sup_{z \in \mathbb{C} \setminus B\left[-\frac{1}{2\varrho_1}, \frac{1}{2\varrho_1}\right]} \|N(z)\| < \frac{1}{C}.$$

Then the evolutionary problem

$$(\partial_0 (M(\partial_0^{-1}) + \partial_0^{-1} N(\partial_0^{-1})) + A) u = f$$

is well-posed and exponentially stable with stability rate ϱ_1 .

Proof. Using the equality

$$zM(z^{-1}) + N(z^{-1}) + A = \left(N(z^{-1}) (zM(z^{-1}) + A)^{-1} + 1 \right) (zM(z^{-1}) + A)$$

we obtain that $zM(z^{-1}) + N(z^{-1}) + A$ is boundedly invertible with

$$\sup_{z \in \mathbb{C} \setminus B\left[-\frac{1}{2\varrho_1}, \frac{1}{2\varrho_1}\right]} \left\| (zM(z^{-1}) + N(z^{-1}) + A)^{-1} \right\| \leq \frac{C}{1 - \|N\|_{\infty} C}.$$

The assertion now follows from Theorem 2.7. \square

3 Second order problems and exponential decay

Frequently, hyperbolic problems occurring in mathematical physics are given as a differential equation of second order in time and space. To obtain an exponential decay one has to assume suitable boundary conditions, making the spatial operator (e.g. the Dirichlet-Laplacian) continuously invertible. Following our solution theory, we have to reformulate the problem as a first order problem. As it turns out, there are several possibilities to do this, allowing to introduce an additional parameter.

We begin to state an exponential stability result for evolutionary equations, where the spatial operator A is assumed to be invertible.

Proposition 3.1. *Let H be a Hilbert space, $A : D(A) \subseteq H \rightarrow H$ m -accretive and continuously invertible and $r > 0$. Moreover, let $M : \mathbb{C} \setminus B[-r, r] \rightarrow L(H)$ be analytic and assume that there exists $\delta \in [0, \frac{1}{2r}[$ such that*

$$K := \sup_{z \in B[0, \delta] \setminus \{0\}} \|zM(z^{-1})\| < \|A^{-1}\|^{-1} \quad (7)$$

and

$$\exists c > 0, 0 < \varrho_0 < \frac{1}{2r} \quad \forall z \in \mathbb{C}_{\Re > -\varrho_0} \setminus B[0, \delta] : \Re zM(z^{-1}) \geq c. \quad (8)$$

Then the evolutionary problem

$$(\partial_0 M(\partial_0^{-1}) + A)u = f$$

is well-posed and exponentially stable.

Proof. By assumption there exist $c > 0$ and $0 < \varrho_0 < \frac{1}{2r}$ such that

$$\Re zM(z^{-1}) \geq c \quad (z \in \mathbb{C}_{\Re > -\varrho_0} \setminus B[0, \delta]).$$

Consequently, $zM(z^{-1}) + A$ is continuously invertible for each $z \in \mathbb{C}_{\Re > -\varrho_0} \setminus B[0, \delta]$ with

$$\|(zM(z^{-1}) + A)^{-1}\| \leq \frac{1}{c}.$$

In particular, this implies that the evolutionary problem is well-posed. Moreover, for $z \in B[0, \delta] \setminus \{0\}$ we have that

$$\|zM(z^{-1})\| \leq K < \|A^{-1}\|^{-1}$$

and hence, we obtain that $zM(z^{-1}) + A$ is continuously invertible for all $z \in B[0, \delta] \setminus \{0\}$ with

$$\|(zM(z^{-1}) + A)^{-1}\| \leq \frac{\|A^{-1}\|}{1 - K\|A^{-1}\|}.$$

Thus, we have that

$$\mathbb{C}_{\Re > -\varrho_0} \setminus \{0\} \ni z \mapsto (zM(z^{-1}) + A)^{-1} \in L(H)$$

is a bounded and analytic mapping and hence, the assertion follows from Theorem 2.7. \square

3.1 Hyperbolic problems

We now start from a second order hyperbolic equation of the form

$$(\partial_0^2 M(\partial_0^{-1}) + C^* C) u = f, \quad (9)$$

where $C : D(C) \subseteq H_0 \rightarrow H_1$ is a densely defined closed linear operator between two Hilbert spaces H_0 and H_1 , which is boundedly invertible, $M(\partial_0^{-1}) = M_0(\partial_0^{-1}) + \partial_0^{-1} M_1(\partial_0^{-1})$, where $M_0, M_1 : \mathbb{C} \setminus B[-r, r] \rightarrow L(H_0)$ are analytic bounded mappings for some $r > 0$. We may rewrite this problem as a first order system in the new unknowns $v := \partial_0 u + du$ and $q := Cu$, where $d > 0$ is an arbitrary parameter. We have that

$$\begin{aligned} \partial_0 M(\partial_0^{-1}) v &= \partial_0^2 M(\partial_0^{-1}) u + d \partial_0 M(\partial_0^{-1}) u \\ &= \partial_0^2 M(\partial_0^{-1}) u + d M_0(\partial_0^{-1}) \partial_0 u + d M_1(\partial_0^{-1}) u \\ &= \partial_0^2 M(\partial_0^{-1}) u + d M_0(\partial_0^{-1}) (v - du) + d M_1(\partial_0^{-1}) u \end{aligned}$$

and, consequently,

$$(\partial_0 M(\partial_0^{-1}) - d M_0(\partial_0^{-1})) v + d (d M_0(\partial_0^{-1}) - M_1(\partial_0^{-1})) C^{-1} q + C^* q = f.$$

Hence, the resulting system reads as

$$\left(\partial_0 \begin{pmatrix} M(\partial_0^{-1}) & 0 \\ 0 & 1 \end{pmatrix} + d \begin{pmatrix} -M_0(\partial_0^{-1}) & (d M_0(\partial_0^{-1}) - M_1(\partial_0^{-1})) C^{-1} \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & C^* \\ -C & 0 \end{pmatrix} \right) \begin{pmatrix} v \\ q \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}. \quad (10)$$

We now consider the new material law $M_d(\partial_0^{-1})$ depending on the additional parameter $d > 0$ induced by the function

$$M_d(z) := \begin{pmatrix} M(z) & 0 \\ 0 & 1 \end{pmatrix} + dz \begin{pmatrix} -M_0(z) & (d M_0(z) - M_1(z)) C^{-1} \\ 0 & 1 \end{pmatrix}, \quad (11)$$

for $z \in \mathbb{C} \setminus B[-r, r]$. The aim is to show that under suitable assumptions on the material law $M(\partial_0^{-1})$, we can show that M_d satisfies the conditions (7) and (8) of Proposition 3.1.

Remark 3.2. Note that if we can show that (10) is exponentially stable, we get that (9) is exponentially stable in the sense that $\partial_0 u, u \in H_{-\nu}(\mathbb{R}; H_0)$ and $Cu \in H_{-\nu}(\mathbb{R}; H_1)$, if $f \in H_{-\nu}(\mathbb{R}; H_0)$. Indeed, the exponential stability of (10) yields $v = \partial_0 u + du \in H_{-\nu}(\mathbb{R}; H_0)$ and $q = Cu \in H_{-\nu}(\mathbb{R}; H_1)$. By the bounded invertibility of C , we read off that $u \in H_{-\nu}(\mathbb{R}; H)$ and hence, $\partial_0 u = v - du \in H_{-\nu}(\mathbb{R}; H)$. Moreover, we note that we also read off a pointwise decay estimate for u by

$$\begin{aligned} |u(t)| &= \left| \int_t^\infty \partial_0 u(s) \, ds \right| \\ &\leq |\partial_0 u|_{H_{-\nu}(\mathbb{R}; H)} e^{-\nu t} \quad (t \in \mathbb{R}), \end{aligned}$$

which yields $|u(t)|^{\mu t} \rightarrow 0$ as $t \rightarrow \infty$ for each $0 < \mu < \nu$.

We begin with the following lemma.

Lemma 3.3. *Let M_d be given as in (11) and let $z \in \mathbb{C} \setminus B[-r, r]$. If there is $c > 0$ such that $\Re z^{-1}M(z) \geq c$ then*

$$\Re z^{-1}M_d(z) \geq \min \left\{ c - dK(d), \frac{3}{4}d + \Re z \right\},$$

where

$$K(d) := \|M_0\|_\infty + (d\|M_0\|_\infty + \|M_1\|_\infty\|C^{-1}\|)^2.$$

Proof. By assumption we have $\Re z^{-1} \geq -\frac{1}{2r}$ and we estimate for $(x, y) \in H_0 \oplus H_1$

$$\begin{aligned} & \Re \left\langle z^{-1}M_d(z) \begin{pmatrix} x \\ y \end{pmatrix} \middle| \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle \\ & \geq c|x|^2 + d\Re \langle -M_0(z)x + (dM_0(z) - M_1(z))C^{-1}y | x \rangle + (d + \Re z)|y|^2 \\ & \geq (c - d\|M_0\|_\infty)|x|^2 - d(d\|M_0\|_\infty + \|M_1\|_\infty\|C^{-1}\|)|x||y| + (d + \Re z)|y|^2 \\ & \geq \left(c - d\|M_0\|_\infty - \frac{1}{4\varepsilon}d^2(d\|M_0\|_\infty + \|M_1\|_\infty\|C^{-1}\|)^2 \right) |x|^2 + (d + \Re z - \varepsilon)|y|^2, \end{aligned}$$

for each $\varepsilon > 0$. Choosing $\varepsilon = \frac{d}{4}$, we obtain the assertion. \square

We begin to treat the case, when the function M satisfies (8) for $\delta = 0$ (note that condition (7) is trivially satisfied for $\delta = 0$).

Proposition 3.4. *Let M_d be given as above and assume that*

$$\exists c > 0 \forall z \in \mathbb{C} \setminus B[-r, r] : \Re z^{-1}M(z) \geq c. \quad (12)$$

Then there exists $d_0 > 0$ such that the function M_{d_0} satisfies (8) for $\delta = 0$.

Proof. Let $z \in \mathbb{C} \setminus B\left[-\frac{1}{2\varrho_0}, \frac{1}{2\varrho_0}\right]$ where $\varrho_0 \in]0, \frac{1}{2r}[$ will be chosen later. Consequently $\Re z^{-1} \geq -\varrho_0$ and we obtain due to Lemma 3.3

$$\Re z^{-1}M_d(z) \geq \min \left\{ c - dK(d), \frac{3}{4}d - \varrho_0 \right\}.$$

Choosing now d_0 small enough such that $c - d_0K(d_0) > 0$ and then choosing $\varrho_0 < \frac{3}{4}d_0$ we derive the positive definiteness constraint (8) for M_{d_0} with $\delta = 0$. \square

Example 3.5. The latter proposition applies to material laws of the form $M(z) = M_0 + zM_1$, where $M_0 \in L(H)$ is a selfadjoint, non-negative operator, $M_1 \in L(H)$ is strictly positive definite and $z \in \mathbb{C}$. Indeed, $M_0(z) = M_0$ and $M_1(z) = M_1$ are trivially analytic and bounded mappings on \mathbb{C} and we have that

$$\Re z^{-1}M(z) = \Re z^{-1}M_0 + \Re M_1 \geq \Re z^{-1}M_0 + c$$

for each $z \in \mathbb{C} \setminus \{0\}$, where $c > 0$ is such that $\Re M_1 \geq c$. Hence, choosing $r < \frac{\|M_0\|}{2c}$ we obtain that (12) holds. The corresponding second order-problem is

$$(\partial_0^2 M_0 + \partial_0 M_1 + C^*C)u = f, \quad (13)$$

which therefore is exponentially stable due to Proposition 3.1. Note that the above equation is not simply an abstract damped wave equation, since M_0 is allowed to have a non-trivial kernel. Indeed, let $H = L_2(\Omega)$ for some bounded domain $\Omega \subseteq \mathbb{R}^n$, $-C^*C = \Delta_D$ the Dirichlet-Laplacian, $M_0 = \chi_{\Omega_1}(m)$ for some $\Omega_1 \subseteq \Omega$ and $M_1 = c > 0$. Then (13) is a combination of the heat equation on $\Omega \setminus \Omega_1$ (as $M_0 = 0$ on this set) and the damped wave equation on Ω_1 (as $M_0 = 1$ on Ω_1). Such classes of degenerated differentail equations have been studied for instance in [3, 4], see also the monograph [8].

Of course, we want to apply Proposition 3.1 to a broader class of hyperbolic differential equations such as integro-differential equations or delay equations. It turns out, that in this case the material law $M(\partial_0^{-1})$ fails to satisfy the condition (12). One way to deal with such equations provides our next result.

Proposition 3.6. *Let M_d be given as in (11) and assume that*

$$\forall \delta > 0 \exists \varrho_0 \in \left] 0, \frac{1}{2r} \right[, c > 0 \forall z \in \mathbb{C}_{\Re > -\varrho_0} \setminus B[0, \delta] : \Re z M(z^{-1}) \geq c. \quad (14)$$

Moreover, we assume that $\lim_{z \rightarrow 0} M_1(z^{-1}) = 0$. Then there exists $d_0 > 0$ such that the material law M_{d_0} satisfies (7) and (8) for a suitable $\delta > 0$, where $A := \begin{pmatrix} 0 & C^ \\ -C & 0 \end{pmatrix}$.*

Proof. As M_0 and M_1 are assumed to be bounded, we obtain that

$$G(d) := \sup_{z \in \mathbb{C} \setminus B[-r, r]} \left\| d \begin{pmatrix} -M_0(z) & (dM_0(z) - M_1(z))C^{-1} \\ 0 & 1 \end{pmatrix} \right\| \rightarrow 0 \quad (d \rightarrow 0).$$

Hence, recalling that $M(z^{-1}) = M_0(z^{-1}) + zM_1(z^{-1})$, we estimate for $z \in \mathbb{C}_{\Re > -\frac{1}{2r}} \setminus \{0\}$

$$\begin{aligned} \|zM_d(z^{-1})\| &\leq \max\{\|zM(z^{-1})\|, |z|\} + G(d) \\ &= \max\{|z|\|M_0\|_\infty + \|M_1(z^{-1})\|, |z|\} + G(d). \end{aligned}$$

Thus, choosing $\delta > 0$ and $d_1 > 0$ small enough we obtain that

$$\sup_{z \in B[0, \delta] \setminus \{0\}} \|zM_d(z^{-1})\| \leq \max \left\{ \delta \|M_0\|_\infty + \sup_{z \in B[0, \delta] \setminus \{0\}} \|M_1(z^{-1})\|, \delta \right\} + G(d_1) < \|A^{-1}\|^{-1}$$

for each $0 < d < d_1$. This shows (7) for M_d where $0 < d < d_1$. According to (14) there is $c > 0$ and $\varrho_0 \in]0, \frac{1}{2r}[$ such that

$$\Re z M(z^{-1}) \geq c$$

for all $z \in \mathbb{C}_{\Re > -\varrho_0} \setminus B[0, \delta]$. Thus, by Lemma 3.3 we have that

$$\Re z M_d(z^{-1}) \geq \min \left\{ c - dK(d), \frac{3}{4}d - \varrho_1 \right\}$$

for every $z \in \mathbb{C}_{\Re > -\varrho_1} \setminus B[0, \delta]$, where $0 < \varrho_1 \leq \varrho_0$. Hence, choosing first $0 < d_0 \leq d_1$ small enough such that $d_0 K(d_0) < c$ and then $\varrho_1 < \frac{3}{4}d_0$ we derive (8) for M_{d_0} . \square

3.2 Integro-differential equations

A class of material laws $M(\partial_0^{-1})$ satisfying the assumptions of Proposition 3.6 arises in the study of hyperbolic integro-differential equations of the form

$$\left(\partial_0^2 (1 - k*)^{-1} + C^* C\right) u = f.$$

Following [22], we consider kernels $k : \mathbb{R}_{\geq 0} \rightarrow L(H_0)$ with the following properties

Hypotheses.

- (a) k is weakly measurable, i.e. for each $x, y \in H_0$, the function $\mathbb{R}_{\geq 0} \ni t \mapsto \langle k(t)x|y \rangle$ is measurable,
- (b) $\mathbb{R}_{\geq 0} \ni t \mapsto \|k(t)\|$ is measurable,⁸
- (c) there exists $\alpha > 0$ such that

$$|k|_{L_1, -\alpha} := \int_0^\infty e^{\alpha t} \|k(t)\| \, dt < 1,$$

- (d) $k(t)$ is selfadjoint for almost every $t \in \mathbb{R}_{\geq 0}$,
- (e) $k(t)k(s) = k(s)k(t)$ for almost every $t, s \in \mathbb{R}_{\geq 0}$.

We consider the material law $M(z) = (1 - \sqrt{2\pi} \widehat{k}(-iz^{-1}))^{-1}$ for $z \in \mathbb{C} \setminus B[-r, r]$, where $r = \frac{1}{2\alpha}$ and

$$\widehat{k}(z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-izt} k(t) \, dt, \quad (z \in \mathbb{C}_{\Im < \alpha})$$

where the integral is meant in the weak sense. The operator $M(\partial_0^{-1})$ is then $(1 - k*)^{-1}$ (see [22] for more details). This material law is of the form $M(z) = M_0(z) + zM_1(z)$ with $M_1(z) = 0$, since $\|M(z)\| \leq \frac{1}{1 - |k|_{L_1, -\alpha}}$ for all $z \in \mathbb{C} \setminus B[-r, r]$. In order to obtain (14) for this material law M one needs to assume a similar estimate for the imaginary part of the Fourier transform \widehat{k} , which reads as follows:

Hypotheses.

- (f) For each $\delta > 0$ there is a function $g : \mathbb{R}_{> -\alpha} \rightarrow \mathbb{R}_{\geq 0}$ continuous at 0 with $g(0) > 0$ such that for each $|t| > \delta, \varrho > -\alpha$ we have the estimate⁹

$$t \Im \widehat{k}(t - i\varrho) \leq -g(\varrho).$$

⁸Note that this follows from (a) if H_0 is separable.

⁹Recall, that for a bounded operator $T \in L(H)$ its imaginary part $\Im T$ is defined as the selfadjoint operator $\frac{1}{2i}(T - T^*)$. Thus, Hypothesis (f) means, using assumption (a), that

$$\left\langle \frac{1}{2i} t \left(\widehat{k}(t - i\varrho) - \widehat{k}(-t - i\varrho) \right) x \middle| x \right\rangle \leq -g(\varrho) |x|^2. \quad (x \in H_0, |t| > \delta, \varrho > -\alpha)$$

We can now prove that under the Hypotheses (a)-(f), that the material law $M(z) = (1 - \sqrt{2\pi} \widehat{k}(-iz^{-1}))^{-1}$ satisfies (14). For doing so, let $\delta > 0$ and $z \in \mathbb{C}_{\Re > -\varrho_0} \setminus [-\delta, \delta]^2$, $z = it + \varrho$ for $|t| > \delta$ and $\varrho \geq -\varrho_0$, where $\varrho_0 \in]0, \alpha[$ will be chosen later. We set

$$D := \left| 1 - \sqrt{2\pi} \widehat{k}(t - i\varrho) \right|^{-1}$$

and note that due to assumption (e) the operator $\widehat{k}(t - i\varrho)$ is normal and thus, D and $1 - \sqrt{2\pi} \widehat{k}(-t - i\varrho) = \left(1 - \sqrt{2\pi} \widehat{k}(t - i\varrho) \right)^*$ commute. We estimate for $x \in H_0$:

$$\begin{aligned} \Re \langle zM(z^{-1})x|x \rangle &= \Re(it + \varrho) \langle \left(1 - \sqrt{2\pi} \widehat{k}(t - i\varrho) \right)^{-1} x|x \rangle \\ &\geq \varrho \Re \langle \left(1 - \sqrt{2\pi} \widehat{k}(t - i\varrho) \right)^{-1} x|x \rangle - t \Im \langle \left(1 - \sqrt{2\pi} \widehat{k}(t - i\varrho) \right)^{-1} x|x \rangle. \\ &= \varrho \Re \langle \left(1 - \sqrt{2\pi} \widehat{k}(-t - i\varrho) \right) Dx|Dx \rangle - t \Im \langle \left(1 - \sqrt{2\pi} \widehat{k}(-t - i\varrho) \right) Dx|Dx \rangle \\ &= \varrho \Re \langle \left(1 - \sqrt{2\pi} \widehat{k}(-t - i\varrho) \right) Dx|Dx \rangle - \sqrt{2\pi} \langle t \Im \widehat{k}(t - i\varrho) Dx|Dx \rangle \\ &\geq \varrho \Re \langle \left(1 - \sqrt{2\pi} \widehat{k}(-t - i\varrho) \right) Dx|Dx \rangle + \sqrt{2\pi} g(\varrho) |Dx|^2. \end{aligned} \quad (15)$$

If $\varrho \in [-\varrho_0, \varrho_0]$ for some $0 < \varrho_0 < \alpha$, the latter term can be estimated by

$$\frac{-\varrho_0 (1 + |k|_{1,-\alpha}) + \sqrt{2\pi} \inf_{\varrho \in [-\varrho_0, \varrho_0]} g(\varrho)}{(1 + |k|_{1,-\alpha})^2} |x|^2,$$

where we have used

$$|x| = |D^{-1}Dx| \leq (1 + |k|_{1,-\alpha}) |Dx|.$$

Since $\frac{-\varrho_0(1+|k|_{1,-\alpha})+\sqrt{2\pi}\inf_{\varrho \in [-\varrho_0, \varrho_0]} g(\varrho)}{(1+|k|_{1,-\alpha})^2} \rightarrow \frac{\sqrt{2\pi}g(0)}{(1+|k|_{1,-\alpha})^2} > 0$ as $\varrho_0 \rightarrow 0$, we find $\varrho_0 \in]0, \alpha[$ such that

$$\frac{-\varrho_0 (1 + |k|_{1,-\alpha}) + \sqrt{2\pi} \inf_{\varrho \in [-\varrho_0, \varrho_0]} g(\varrho)}{(1 + |k|_{1,-\alpha})^2} > 0.$$

If $\varrho > \varrho_0$ we have that (15) can be estimated by

$$\frac{\varrho (1 - |k|_{1,-\alpha}) + \sqrt{2\pi} g(\varrho)}{(1 + |k|_{1,-\alpha})^2} |x|^2 \geq \frac{\varrho_0 (1 - |k|_{1,-\alpha})}{(1 + |k|_{1,-\alpha})^2} |x|^2.$$

Summarizing, we have shown that (14) holds for our material law M and hence, the corresponding evolutionary equation

$$\left(\partial_0^2 (1 - k^*)^{-1} + C^* C \right) u = f$$

is exponentially stable by Proposition 3.6.

Remark 3.7. In [2] the authors consider scalar-valued kernels $k \in L_{1,-\alpha}(\mathbb{R}_{\geq 0}; \mathbb{R})$ for some $\alpha > 0$, such that $t \mapsto \int_t^\infty e^{\alpha s} k(s) ds$ defines a strongly positive definite kernel on $L_{2,\text{loc}}(\mathbb{R}_{\geq 0})$ and $\int_0^\infty e^{\alpha s} |k(s)| ds < 1$. Clearly, these kernels satisfy the hypotheses (a)-(e). Moreover, following [2, Proposition 2.2, Proposition 2.5] these kernels satisfy an estimate of the form

$$\exists c > 0 \forall t > 0, \varrho > -\alpha : \int_0^\infty \sin(ts) e^{-\varrho s} k(s) dt \geq c \frac{1}{(\alpha + \varrho + 1)^2} \frac{t}{1 + t^2}.$$

Hence,

$$\begin{aligned}
t \Im \widehat{k}(t - i\rho) &= t \frac{1}{\sqrt{2\pi}} \int_0^\infty \sin(-ts) e^{-\rho s} k(s) \, ds \\
&= -t \frac{1}{\sqrt{2\pi}} \int_0^\infty \sin(ts) e^{-\rho s} k(s) \, dt \\
&\leq -\frac{c}{\sqrt{2\pi}(\alpha + \rho + 1)^2} \frac{t^2}{1 + t^2}
\end{aligned}$$

for each $t \geq 0, \rho > -\alpha$. Since $\Im \widehat{k}(-t - i\rho) = -\Im \widehat{k}(t - i\rho)$, we obtain

$$t \Im \widehat{k}(t - i\rho) \leq -\frac{c}{\sqrt{2\pi}(\alpha + \rho + 1)^2} \frac{t^2}{1 + t^2}$$

for every $t \in \mathbb{R}, \rho > -\alpha$. Let $\delta > 0$ and $|t| > \delta$. Then

$$t \Im \widehat{k}(t - i\rho) \leq -\frac{c}{\sqrt{2\pi}(\alpha + \rho + 1)^2} \frac{\delta^2}{1 + \delta^2}$$

for $\rho > -\alpha$. As $g = \left(\rho \mapsto \frac{c}{\sqrt{2\pi}(\alpha + \rho + 1)^2} \frac{\delta^2}{1 + \delta^2} \right)$ is continuous and attains positive values only, we have that k satisfies (f) and thus, the exponential stability of the corresponding evolutionary equation is covered by our theory.

4 The wave equation with a time delay and a convolution integral

Motivated by a recent paper of Alabau-Boussouira et al. [1], we study the following wave equation

$$\partial_0^2 u - (1 - k*)\Delta u + \kappa \tau_{-h} \partial_0 u = f \quad (16)$$

on a domain $\Omega \subseteq \mathbb{R}^n$ with homogeneous Dirichlet boundary conditions, i.e.

$$u = 0 \text{ on } \partial\Omega. \quad (17)$$

Here τ_{-h} denotes the translation operator for some $h > 0$, i.e. $(\tau_{-h}f)(t) = f(t - h)$ and $\kappa \in \mathbb{R}$ is a given parameter. In [1], the kernel $k : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is assumed to be locally absolutely continuous, $k(0) > 0$, $\int_0^\infty k(s) \, ds < 1$ and $k'(t) \leq -\alpha k(t)$ for some $\alpha > 0$ and each $t > 0$. We will generalize this to operator-valued kernels satisfying the assumptions given in Subsection 3.2. Moreover, we will show that (16) fits into our general setting and that the exponential stability can be shown under the assumption that $|\kappa|$ is sufficiently small. This is exactly the result stated in [1], however under weaker assumptions on the kernel k and by a completely different approach. First of all, we show that (16) is indeed of the form (9). For doing so, we need to introduce the spatial differential operators involved.

Definition. Let $\Omega \subseteq \mathbb{R}^n$. We define the gradient with vanishing boundary values grad_c as the closure of

$$\begin{aligned} \text{grad}_c|_{C_c^\infty(\Omega)} : C_c^\infty(\Omega) &\subseteq L_2(\Omega) \rightarrow L_2(\Omega)^n \\ \phi &\mapsto (\partial_i \phi)_{i \in \{1, \dots, n\}}. \end{aligned}$$

Moreover, we define $\text{div} := -\text{grad}_c^*$, the divergence with maximal domain in $L_2(\Omega)^n$.

Remark 4.1. We note that by definition, the domain of grad_c is nothing but the well-known Sobolev-space $H_0^1(\Omega)$ – the closure of the test function $C_c^\infty(\Omega)$ with respect to the topology on $H^1(\Omega)$. In consequence, the domain of div is

$$D(\text{div}) = \left\{ \Phi = (\Phi_i)_{i \in \{1, \dots, n\}} \in L_2(\Omega)^n \mid \text{div } \Phi = \sum_{i=1}^n \partial_i \Phi_i \in L_2(\Omega) \right\},$$

where $\partial_i \Phi_i$ is meant in the distributional sense.

Using these operators, (16) together with the boundary condition (17) reads as

$$\partial_0^2 u - (1 - k*) \text{div grad}_c u + \kappa \tau_{-h} \partial_0 u = f. \quad (18)$$

From now on we assume that the kernel k is operator-valued, i.e. $k : \mathbb{R}_{\geq 0} \rightarrow L(L_2(\Omega))$, and satisfies the hypotheses (a)-(f) of Subsection 3.2. The next lemma shows that this is indeed a generalization of the assumptions on k made in [1].

Lemma 4.2. *Let $k : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ be locally absolutely continuous, $\int_0^\infty k(s) ds < 1$, $k(0) > 0$ and $k'(t) \leq -\alpha_0 k(t)$ for some $\alpha_0 > 0$ and each $t \in \mathbb{R}_{\geq 0}$. Then k satisfies (a)-(f) in Subsection 3.2.*

Proof. The Hypotheses (a),(b),(d) and (e) are trivially satisfied. Moreover, we observe that $k'(t) \leq -\alpha_0 k(t)$ yields that $k(t) \leq k(0)e^{-\alpha_0 t}$ for each $t \in \mathbb{R}_{\geq 0}$ and thus,

$$\int_0^\infty k(t) e^{\alpha t} dt < \infty \quad (\alpha < \alpha_0).$$

Since

$$\mathbb{R}_{<\alpha_0} \ni \alpha \mapsto \int_0^\infty k(t) e^{\alpha t} dt$$

is continuous, it follows from $\int_0^\infty k(s) ds < 1$ that (c) holds. Moreover, we have that

$$\int_0^N e^{\alpha t} |k'(t)| dt = - \int_0^N e^{\alpha t} k'(t) dt = -e^{\alpha N} k(N) + k(0) + \alpha \int_0^N e^{\alpha t} k(t) dt \quad (\alpha < \alpha_0, N \in \mathbb{N}),$$

which gives $k' \in L_{1,-\alpha}(\mathbb{R}_{\geq 0})$, as the latter term converges to $k(0) + \alpha |k|_{1,-\alpha}$ as N tends to infinity. It is left to show that k satisfies Hypothesis (f) of Subsection 3.2. Let $0 < \alpha < \alpha_0$. We claim that

$$\exists c > 0 \forall t > 0 : \Im \widehat{k}(t + i\alpha) \leq -c \frac{t}{1 + t^2}. \quad (19)$$

Let us consider the function

$$\Phi(t) := -\frac{1+t^2}{t} \Im \widehat{k}(t + i\alpha) = \frac{1}{\sqrt{2\pi}} \frac{1+t^2}{t} \int_0^\infty \sin(ts) e^{\alpha s} k(s) \, ds \quad (t > 0).$$

Obviously, Φ is continuous. Moreover, $\Phi(t) > 0$ for each $t > 0$. Indeed, we compute

$$\begin{aligned} \int_0^\infty \sin(ts) e^{\alpha s} k(s) \, ds &= \frac{1}{t} \left(\int_0^\infty \cos(ts) (\alpha k(s) + k'(s)) e^{\alpha s} \, ds + k(0) \right) \\ &= \frac{1}{t} \int_0^\infty (1 - \cos(ts)) (-\alpha k(s) - k'(s)) e^{\alpha s} \, ds \\ &\geq \frac{1}{t} (\alpha_0 - \alpha) \int_0^\infty (1 - \cos(ts)) k(s) e^{\alpha s} \, ds \end{aligned}$$

and since the integrand in the latter integral is positive (except for $ts = (2j+1)\frac{\pi}{2}$, $j \in \mathbb{N}$), we derive that $\Phi(t) > 0$. Moreover, using the latter computation we have that

$$\begin{aligned} \Phi(t) &\geq (\alpha_0 - \alpha) \frac{1+t^2}{t^2} \frac{1}{\sqrt{2\pi}} \int_0^\infty (1 - \cos(ts)) k(s) e^{\alpha s} \, ds \\ &= (\alpha_0 - \alpha) \frac{1+t^2}{t^2} \left(\frac{1}{\sqrt{2\pi}} |k|_{1,\alpha} - \Re \widehat{k}(t + i\alpha) \right). \end{aligned}$$

Thus, by the Riemann-Lebesgue Lemma we get $\liminf_{t \rightarrow \infty} \Phi(t) \geq \frac{(\alpha_0 - \alpha)}{\sqrt{2\pi}} |k|_{1,\alpha} > 0$. Furthermore, using the rule of l'Hospital, we compute

$$\begin{aligned} \lim_{t \rightarrow 0} \Phi(t) &= \lim_{t \rightarrow 0} \frac{1}{t} \frac{1}{\sqrt{2\pi}} \int_0^\infty \sin(ts) e^{\alpha s} k(s) \, ds \\ &= \lim_{t \rightarrow 0} \frac{1}{\sqrt{2\pi}} \int_0^\infty \cos(ts) s e^{\alpha s} k(s) \, ds \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty s e^{\alpha s} k(s) \, ds > 0, \end{aligned}$$

where we have used that $(s \mapsto s e^{\alpha s} k(s)) \in L_1(\mathbb{R}_{\geq 0})$, which follows since $k(s) \leq k(0) e^{-\alpha_0 s}$ for $s \geq 0$. Summarizing, we have shown that $\Phi : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$ is a continuous function with $\lim_{t \rightarrow 0} \Phi(t) > 0$ and $\liminf_{t \rightarrow \infty} \Phi(t) > 0$ and hence, there is a constant $c > 0$ with $\Phi(t) \geq c$ for each $t > 0$. This proves (19). Finally, by [2, Proposition 2.5], estimate (19) implies

$$\forall t > 0, \varrho > -\alpha : \Im \widehat{k}(t - i\varrho) \leq -\frac{c}{(1 + \alpha + \varrho)^2} \frac{t}{1 + t^2},$$

which yields (f) (cp. Remark 3.7). □

We come back to the exponential stability of (18). Since we have assumed that k satisfies Hypothesis (c), we get that $(1 - k*)$ is a boundedly invertible operator on $H_\varrho(\mathbb{R}; L_2(\Omega))$ for each $\varrho \geq -\alpha$. Consequently, (18) may be written as¹⁰

$$\left(\partial_0^2 (1 - k*)^{-1} - \operatorname{div} \operatorname{grad}_c + \kappa (1 - k*)^{-1} \tau_{-h} \partial_0 \right) u = (1 - k*)^{-1} f =: \tilde{f}. \quad (20)$$

We assume that grad_c is injective and has closed range, which can for instance be achieved by assuming that Ω is bounded due to Poincaré's inequality. We denote by ι the canonical embedding of $R(\operatorname{grad}_c)$ into $L_2(\Omega)^n$, i.e.

$$\begin{aligned} \iota : R(\operatorname{grad}_c) &\rightarrow L_2(\Omega)^n \\ F &\mapsto F. \end{aligned}$$

Consequently, $\iota^* : L_2(\Omega)^n \rightarrow R(\operatorname{grad}_c)$ is the orthogonal projection onto the space $R(\operatorname{grad}_c)$ and we get $-\operatorname{div} \operatorname{grad}_c = -\operatorname{div} \iota^* \operatorname{grad}_c$. Setting $C : D(\operatorname{grad}_c) \subseteq L_2(\Omega) \rightarrow R(\operatorname{grad}_c)$ defined by $C = -\iota^* \operatorname{grad}_c$ we get that C is boundedly invertible by the closed graph theorem and $C^* = \operatorname{div} \iota$ (see e.g. [23, Lemma 2.4]). Hence, (20) is of the form (9) with

$$\begin{aligned} M_0(z) &= \left(1 - \sqrt{2\pi} k(-iz^{-1}) \right)^{-1}, \\ M_1(z) &= \kappa \left(1 - \sqrt{2\pi} \widehat{k}(-iz^{-1}) \right)^{-1} e^{-hz^{-1}}, \\ C &= \iota^* \operatorname{grad}_c. \end{aligned}$$

Unfortunately, the material law $M(\partial_0^{-1}) = M_0(\partial_0^{-1}) + \partial_0^{-1} M_1(\partial_0^{-1})$ does not satisfy the assumption of Proposition 3.4 or Proposition 3.6. But we can prove the exponential stability by a perturbation argument. Let us first recall, how to write (20) as a first order system. We define the material law M_d for some $d > 0$ by

$$\begin{aligned} M_d(z) &= \begin{pmatrix} M(z) & 0 \\ 0 & 1 \end{pmatrix} + dz \begin{pmatrix} -M_0(z) & (dM_0(z) - M_1(z)) C^{-1} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} M_0(z) & 0 \\ 0 & 1 \end{pmatrix} + dz \begin{pmatrix} -M_0(z) & (dM_0(z)) C^{-1} \\ 0 & 1 \end{pmatrix} + z \left(\begin{pmatrix} M_1(z) & 0 \\ 0 & 0 \end{pmatrix} - d \begin{pmatrix} 0 & M_1(z) C^{-1} \\ 0 & 0 \end{pmatrix} \right) \end{aligned} \quad (21)$$

and set

$$\begin{aligned} \widetilde{M}_d(z) &:= \begin{pmatrix} M_0(z) & 0 \\ 0 & 1 \end{pmatrix} + dz \begin{pmatrix} -M_0(z) & (dM_0(z)) C^{-1} \\ 0 & 1 \end{pmatrix}, \\ N_d(z) &:= \begin{pmatrix} M_1(z) & 0 \\ 0 & 0 \end{pmatrix} - d \begin{pmatrix} 0 & M_1(z) C^{-1} \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Thus, the first order formulation of (20) can be written as

$$\left(\partial_0 \widetilde{M}_d(\partial_0^{-1}) + N_d(\partial_0^{-1}) + \begin{pmatrix} 0 & C^* \\ -C & 0 \end{pmatrix} \right) \begin{pmatrix} v \\ q \end{pmatrix} = \begin{pmatrix} \tilde{f} \\ 0 \end{pmatrix}, \quad (22)$$

where $v := \partial_0 u + du$ and $q := Cu$. We note that according to our findings in Subsection 3.2 and by Proposition 3.1, there is $d > 0$ such that

$$\left(\partial_0 \widetilde{M}_d(\partial_0^{-1}) + \begin{pmatrix} 0 & C^* \\ -C & 0 \end{pmatrix} \right) \begin{pmatrix} v \\ q \end{pmatrix} = \begin{pmatrix} \tilde{f} \\ 0 \end{pmatrix}$$

¹⁰Note that $\tilde{f} \in H_\varrho(\mathbb{R}; L_2(\Omega))$ if $f \in H_\varrho(\mathbb{R}; L_2(\Omega))$.

is exponentially stable, since the kernel k is assumed to satisfy the Hypotheses (a)-(f). We denote its stability rate by ϱ_1 . Moreover, since

$$\|N_d(z)\| \leq \sqrt{1 + d^2\|C^{-1}\|^2} \|M_1(z)\| \leq \sqrt{1 + d^2\|C^{-1}\|^2} |\kappa| (1 - |k|_{1,-\varrho_1})^{-1} e^{h\varrho_1},$$

for each $z \in \mathbb{C} \setminus B\left[-\frac{1}{2\varrho_1}, \frac{1}{2\varrho_1}\right]$, we may choose $|\kappa|$ small enough to get

$$\sup_{z \in \mathbb{C} \setminus B\left[-\frac{1}{2\varrho_1}, \frac{1}{2\varrho_1}\right]} \|N_d(z)\| < \sup_{z \in \mathbb{C} \setminus B\left[-\frac{1}{2\varrho_1}, \frac{1}{2\varrho_1}\right]} \left\| \left(z^{-1} \widetilde{M}_d(z) + \begin{pmatrix} 0 & C^* \\ -C & 0 \end{pmatrix} \right)^{-1} \right\|.$$

Then, by Corollary 2.9 we derive the exponential stability of (22). We summarize our findings in the following theorem.

Theorem 4.3. *Let $\Omega \subseteq \mathbb{R}^n$ be a domain, such that the Poincaré inequality holds, i.e.*

$$\exists c > 0 \forall u \in D(\text{grad}_c) : |u|_{L_2(\Omega)} \leq c |\text{grad}_c u|_{L_2(\Omega)^n}.$$

Moreover, let $k : \mathbb{R}_{\geq 0} \rightarrow L(L_2(\Omega)^n)$ be a kernel that satisfies the Hypotheses (a)-(f) of Subsection 3.2 and let $h > 0$. Then there exists a $\kappa_0 > 0$ such that for each $|\kappa| < \kappa_0$ the problem

$$\partial_0^2 u - (1 - k*) \text{div grad}_c u + \kappa \tau_{-h} \partial u = f$$

is exponentially stable (cp. Remark 3.2).

5 Acknowledgement

We thank Marcus Waurick for carefully reading the text and for fruitful discussions, especially about the proof of Theorem 2.7.

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